

Six-dimensional spaces defined by the equations KN and KdV

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Abstract

On base of three-dimensional flat metrics obtained with the help of solutions of the KdV-equation were constructed the examples of six-dimensional metrics, which are determined by the help of solutions of the Krichever-Novikov equation and the KdV equation. Their properties are discussed.

1 Flat $6D$ -spaces

Theorem 1 *The metric of the $6D$ -space in local coordinates $x^i = (x, y, z, u, v, w)$*

$$ds^2 = y^2 dx^2 + 2 \left(y^2 l(x, z) - 1/2 \right) dx dz + 2 dy dz + \left(y^2 (l(x, z))^2 - 2y \frac{\partial}{\partial x} l(x, z) + l(x, z) \right) dz^2 +$$

$$+ 2 \frac{B(x, y, z) du dv}{(wu + vw + uv)^2} + 2 \frac{B(x, y, z) du dw}{(wu + vw + uv)^2} + 2 \frac{B(x, y, z) dv dw}{(wu + vw + uv)^2} \quad (1)$$

is a flat if the following conditions on the coefficients hold

$$B(x, y, z) = 1/4 (F_1(x, z))^2 y^2 + 1/2 F_1(x, z) y F_2(x, z) + 1/4 (F_2(x, z))^2$$

$$F_1(x, z) = -2 \frac{\partial}{\partial x} F_2(x, z),$$

$$l(x, z) = -1/3 \frac{\frac{\partial}{\partial z} F_2(x, z) - 2 \frac{\partial^3}{\partial x^3} F_2(x, z)}{\frac{\partial}{\partial x} F_2(x, z)}, \quad (2)$$

where the function $F_2(x, z)$ satisfies the equation

$$\frac{\partial}{\partial z} F_2(x, z) = -\frac{\partial^3}{\partial x^3} F_2(x, z) + 3/2 \frac{\left(\frac{\partial^2}{\partial x^2} F_2(x, z) \right)^2}{\frac{\partial}{\partial x} F_2(x, z)}, \quad (3)$$

which is the particular case of the Krichever-Novikov equation integrable by the IST-method [1].

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This theorem is generalization of the following result obtained by the author [2]

Theorem 2 *The metric of the 3D-space in local coordinates $x^i = (x, y, z)$*

$$ds^2 = y^2 dx^2 + 2 \left(y^2 l(x, z) - 1/2 \right) dx dz + 2 dy dz + \left(y^2 (l(x, z))^2 - 2 y \frac{\partial}{\partial x} l(x, z) + l(x, z) \right) dz^2 \quad (4)$$

is a flat if the function $l(x, z)$ satisfies the KdF-equation

$$\frac{\partial}{\partial z} l(x, z) - 3 l(x, z) \frac{\partial}{\partial x} l(x, z) + \frac{\partial^3}{\partial x^3} l(x, z) = 0. \quad (5)$$

Both equations can be of used to study the properties of the six-dimensional metrics having applications in the theory of Calaby-Yau spaces.

Theorem 3 *Six-dimensional space in local coordinates $x^i = (x, y, z, u, v, w)$ equipped with the metric*

$$\begin{aligned} ds^2 = & y^2 dx^2 + 2 \left(y^2 l(x, z) + m(x, z) \right) dx dz + \\ & + 2 dy dz + \left(y^2 (l(x, z))^2 - 2 y \frac{\partial}{\partial x} l(x, z) + 2 l(x, z) + 2 l(x, z) m(x, z) \right) dz^2 + \\ & + v^2 du^2 + 2 \left(v^2 L(u, w) + M(u, w) \right) du dw + 2 dv dw + \\ & + \left(v^2 (L(u, w))^2 - 2 v \frac{\partial}{\partial u} L(u, w) + 2 L(u, w) + 2 L(u, w) M(u, w) \right) dw^2 \end{aligned} \quad (6)$$

with the coefficients

$$l(x, z) = \left(-1/2 \frac{\left(\frac{\partial^2}{\partial x^2} F_2(x, z) \right)^2}{\frac{\partial}{\partial x} F_2(x, z)} + \frac{\partial^3}{\partial x^3} F_2(x, z) \right) \left(\frac{\partial}{\partial x} F_2(x, z) \right)^{-1},$$

$$m(x, z) = -1/2, \quad F_1(x, z) = -2 \frac{\partial}{\partial x} F_2(x, z)$$

and with condition

$$\frac{\partial}{\partial w} M(u, w) = L(u, w) \frac{\partial}{\partial u} M(u, w) + \frac{\partial}{\partial u} L(u, w) + 2 \left(\frac{\partial}{\partial u} L(u, w) \right) M(u, w)$$

is a flat if the function $F_2(x, z)$ satisfies the KN-equation

$$\frac{\partial}{\partial z} F_2(x, z) + \frac{\partial^3}{\partial x^3} F_2(x, z) - 3/2 \frac{\left(\frac{\partial^2}{\partial x^2} F_2(x, z) \right)^2}{\frac{\partial}{\partial x} F_2(x, z)} = 0,$$

and the function $L(u, w)$ satisfies the KdV-equation

$$\frac{\partial}{\partial w} L(u, w) - 3 L(u, w) \frac{\partial}{\partial u} L(u, w) + \frac{\partial^3}{\partial u^3} L(u, w) = 0.$$

Remark 1 According results of the authors [3] solutions of the KN-equation can be produced by the formula

$$(F_{2j+1})_x = \frac{(F_{2j})^2}{(F_{2j})_x}, \quad j = 0, 1, 2, \dots,$$

in which the function $F_{20}(x, z)$ is the some initial solution of the equation (3).

For example for the simplest function $F_{20} = x$ we get the solution

$$F_{21}(x, z) = 1/3 x^3 + 4z.$$

On the next step we find that the function

$$F_{22}(x, z) = 1/45 x^5 + 4/3 x^2 z - 16 \frac{z^2}{x}$$

is the solution of the KN-equation and so on.

In result we obtain sets of the six-dimensional flat spaces which are determined by solutions of the equations KdF and KN. Some of them can be compacts and from this point of view they may be of used to further applications.

2 Ricci-flat curved 6D-space

Give an example of a 6-dimensional Ricci-flat space $R_{ik} = 0$ with nonzero Riemann tensor $R_{ijkl} \neq 0$.

With this aim we use the construction of Riemann extensions of the space with connection [4].

For the E^n -dimensional space in local coordinates $x^j = (x^1, x^2, \dots, x^n)$ with symmetric coefficients of connexion Π_{ij}^k can be constructed space of the Riemann double dimensionality $D = 2n$ having the metrics of the form

$$ds^2 = -2\Pi_{ij}^k \xi^k dx^i dx^j + 2\xi_k dx^k, \quad (7)$$

where ξ_k -are additive coordinates.

We use this fact for construction of the 6D- space with the Ricci- flat metric on solutions of the KdF-equation but nonzero curvature.

Coefficients of connection of the of the space N^3 with the metric (4) have the form

$$\begin{aligned} \Pi_{11}^1 &= 1/2 \frac{2l(x, z)y^2 - 1}{y}, \quad \Pi_{11}^2 = -1/4 \frac{4y^3 \frac{\partial}{\partial x} l(x, z) - 8l(x, z)y^2 + 1}{y}, \\ \Pi_{11}^3 &= -y, \quad \Pi_{12}^1 = y^{-1}, \quad \Pi_{12}^2 = 1/2 y^{-1} \Pi_{13}^1 = 1/2 \frac{(2l(x, z)y^2 - 1)l(x, z)}{y}, \\ \Pi_{13}^2 &= -1/4 \frac{4y^3 l(x, z) \frac{\partial}{\partial x} l(x, z) - 8y^2 (l(x, z))^2 + l(x, z) + 4y^2 \frac{\partial^2}{\partial x^2} l(x, z) - 2y \frac{\partial}{\partial x} l(x, z)}{y}, \\ \Pi_{13}^3 &= -l(x, z)y, \quad \Pi_{23}^1 = \frac{l(x, z)}{y}, \quad \Pi_{23}^2 = -1/2 \frac{2y \frac{\partial}{\partial x} l(x, z) - l(x, z)}{y}, \\ \Pi_{33}^1 &= -1/2 \frac{-2 \left(\frac{\partial}{\partial z} l(x, z) \right) y + 4yl(x, z) \frac{\partial}{\partial x} l(x, z) - 2 \frac{\partial^2}{\partial x^2} l(x, z) - 2 (l(x, z))^3 y^2 + (l(x, z))^2}{y}, \end{aligned}$$

$$\begin{aligned}
4y\Pi_{33}^2 &= -4(l(x,z))^2 y^3 \frac{\partial}{\partial x} l(x,z) - 4l(x,z)y^2 \frac{\partial^2}{\partial x^2} l(x,z) + 4\left(\frac{\partial}{\partial z} l(x,z)\right)y + 2\frac{\partial^2}{\partial x^2} l(x,z) + \\
&+ 8y^2 \left(\frac{\partial}{\partial x} l(x,z)\right)^2 + 8(l(x,z))^3 y^2 - 8yl(x,z) \frac{\partial}{\partial x} l(x,z) - (l(x,z))^2 - 4y^2 \frac{\partial^2}{\partial x \partial z} l(x,z), \\
\Pi_{33}^3 &= -(l(x,z))^2 y + \frac{\partial}{\partial x} l(x,z).
\end{aligned}$$

The metrics of 6D-space in local coordinates $x^i = (x, y, z, u, v, w)$ in considered case is of the form

$$\begin{aligned}
ds^2 &= -2\Pi_{11}^1 udx^2 - 2\Pi_{33}^1 udz^2 - 2\Pi_{11}^3 wdx^2 - 2\Pi_{11}^2 vdx^2 - 2\Pi_{33}^3 wdz^2 - 2\Pi_{33}^2 vdz^2 - \\
&- 4\Pi_{13}^3 wdx dz - 4\Pi_{12}^1 udx dy - 4\Pi_{12}^2 vdx dy - 4\Pi_{13}^1 udx dz - 4\Pi_{12}^3 vdx dz - 4\Pi_{23}^2 vdy dz - \\
&- 4\Pi_{23}^1 udy dz + 2dudx + 2dvdy + 2dwdz.
\end{aligned} \tag{8}$$

The metric (8) is a flat $R_{ijkl} = 0$ on solution of the KdF-equation (5).

Simple modification of the metric $ds^2 = (8)$ leads to the following statement

Theorem 4 *The metric*

$$\tilde{ds}^2 = ds^2 + \epsilon dy^2$$

is the Richi -flat ($R_{ik} = 0$) on solution of the KdF-equation (5) but it has the nonzero curvature $R_{ijkl} \neq 0$.

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